

NOTES
ON
THE CONSTRUCTION
OF
A SIMPLE SRUTI-SVARA-GRAMA YANTRA*

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THE *rāga* system of music which is prevalent in the classical traditions of both North and South India is a species of that genus of music which we generally call "modal" music. That is to say, it is a system of music based not upon the laws of "Harmony" as those laws came to be discovered in Europe in which a basic scale type is subject to modulations in various keys; rather, it is a system which is based on selected fixed scale types whose base note, tonic or *ṣaḍja*, remains both unchanged and continually sounded throughout a given performance. Each note in the scale of a *rāga* derives its emotive significance with reference to the fixed drone of the tonic, primarily, and secondarily, with reference to its neighboring tones, and indeed, then in relation to all of the notes of the scale.

Since neither the tonic of a *rāga*'s scale type, nor the scale itself can change once the performance of the *rāga* has begun, and since the musician and the listener both have continually in front of them the droning tonic, the *rāga* system of music encourages a precision in intonation which is unknown to Western ears. The harmonic system of music in the Western classical tradition is based upon the division of the octave into twelve tones whose frequencies are logarithmically equidistant from each other. This system of tuning known as "equal temperament" distributes an error of roughly one part in eighty, the comma, which arises from the difference between twelve successive fifths and seven octaves, through the twelve notes of the scale so that each interval of the scale, except the octave interval, is just slightly out of tune. This compromise with Nature which some view as insignificant, and others view as disastrous, allows one to modulate freely on

* An "interval-note-and-scale-calculator".

keyboard or fretted instruments through the twelve keys with no key being any more out of tune than any other key.

In spite of the fact that the introduction of the Harmonium, whose tuning is based upon the system of equal temperament, has tended to influence somewhat the concept of intonation in Indian music, there is no need in the Indian system of music to resort to the use of equal tempered scales. That Paṇḍit Bhāṭkhaṇḍe, whose contributions to the systematic study of Indian music were immense, thought that Indian music was based upon the tempered scale only illustrates the degree to which knowledge of the ancient system of the division of the octave into 22 intervals (*śruti*-s) had failed to be preserved.

This is not the place to document this loss of knowledge, nor its rediscovery which is a fascinating study in its own right. Suffice it to say that the *Saṅgītaratnākara* (c. 13th Century A.D.) is the last Sanskrit text to give a clear exposition of the ancient system. It has taken several generations of scholars, both Western and Indian, the period of the past two hundred years to piece the threads of the ancient system back together, and the job is by no means finished.

Not only did scholars have to arrive at an understanding of what, exactly, the twenty-two *śrutis* are, what their ratios are, and what their frequencies are, given a base *ṣaḍja*, but also the burden has fallen to the lot of modern musicologists both to attempt to reconstruct the ancient system, and to interpret or extend its ideas to modern practice which has changed from ancient practice.

The first job, determining the values of the *śrutis* has pretty much been accomplished. I say this in

spite of the fact that this topic is still considered by some to be moot. The difficulty arises from the fact that the contributions to the problem are widely scattered in books and journals which are out of print and difficult to obtain. Also, a substantial amount has been done through the medium of English which apparently has eluded the eyes of some important scholars using the Hindi medium. Unfortunately, there seems to be suprisingly little cross-fertilization between these two groups of scholars.

For our purposes, we take the *śrutis* to be determineable, following the majority opinion of recent English language writers according to the following system. Take a given note as the base *ṣaḍja* (we take "C" of the Western terminology for the sake of simplicity). Then tune as follows by perfect fifths and—to stay within the region of one octave—perfect fourths :

1. From Sa (C) tune Pa (G).
2. From Pa (G) tune Re (D).
3. From Re (D) tune Dha (A).
4. From Dha (A) tune Ga (E).
5. From Ga (E) tune Ni (B).
6. From Ni (B) tune Ma *tivra* (F).
7. From Ma *tivra* (F) tune Re *komal* (D^b).
8. From Re *komal* (D^b) tune Dha *komal* (A^b).
9. From Dha *komal* (A^b) tune Ga *komal* (E^b).
10. From Ga *komal* (E^b) tune Ni *komal* (B^b).
11. From Ni *komal* (B^b) tune Ma *komal* (F).

If we should attempt to tune by another fifth from Ma *komal* (F) to *ṣaḍja* (C) we find that we reach to a Sa (C) which is a comma higher than our original *ṣaḍja* or base. The twelve notes tuned in this way are tuned according to the cycle of fifths, and because the twelfth step (note 13) introduces a new base a comma higher than our original, eleven steps is considered to mark one complete round in the cycle of fifths. This determines the positions of twelve notes, or as we should say, twelve *śrutis*.

If the actual mathematical ratios of the intervals is of no concern to us (supposing we are tuning strings or reeds) one round of the cycle of fifths is sufficient to establish twelve of the twenty-two *śrutis*. If, on the other hand, we do care about simplicity of ratios, it is necessary to replace the intervals of the last five steps by intervals determined as follows : replace the

Dha *komal* (A^b) obtained by the cycle of fifths by a Dha *komal* which is beneath the base note *ṣaḍja* (C) by the same interval that the fifth harmonic of the base is higher than the fourth harmonic of the base—a "just" major third (5/4). Then from this interval "readjust" the remaining four intervals in the last set of five intervals by tuning perfect fifths (or fourths).

In actual practice, the difference between these replacement intervals and the originals is less than one part in six-hundred. Nevertheless, the difference in simplicity of ratios is striking. For example, the Ma (F) achieved by the cycle of fifths has the ratio 177147/131072, while the Ma (F) achieved by starting from the secondary base, Dha *komal* (A^b) has the ratio 27/20. Because it is possible to match every interval in the cycle of fifths from Sa by an interval determined from the secondary base, Dha *komal*, it is possible to determine a series of *śrutis* adjacent, within one part in six-hundred, to that series which is determineable straight off from the cycle of fifths. Following usual practice, I call those determined by the cycle of fifths "Pythagorean" and those determined from the secondary base by fifths and fourths "Harmonic."

Mixing Pythagorean and Harmonic determinations, it is possible to select a set of twelve intervals (*śrutis*) whose ratios are fairly simple.

Twelve positions are established by the cycle of fifths, or their Harmonic replacements. Likewise, another set of twelve *śruti* positions, eleven really since Sa (C) should be common to both sets, can be determined from the cycle of fourths (the cycle of descending fifths). For this set, tune as follows :

1. From Sa (C) tune Ma *komal* (F).
2. From Ma *komal* tune Ni *komal* (B^b).
3. From Ni *komal* (B^b) tune Ga *komal* (E^b).
4. From Ga *komal* (E^b) tune Dha *komal* (A^b).
5. From Dha *komal* (A^b) tune Re *komal* (D^b).
6. From Re *komal* (D^b) tune Pa *komal* (=Ma *tivra*) (G^b).
7. From Pa *komal* (G^b) tune Ni (B).
8. From Ni (B) tune Ga (E).
9. From Ga (E) tune Dha (A).
10. From Dha (A) tune Re (D).
11. From Re (D) tune Pa (G).

As before, for the sake of simpler ratios, the last five intervals obtained from this cycle of fourths can be replaced by intervals determined as previously but from the major third (5/4) as the secondary base.

Combining these two sets of intervals selecting the simpler ratios gives a set of twenty-three note-positions in which every note except the base, Sa, has two alternative positions separated by a comma's tonal distance. It would take too long here "prove" that this system is equivalent to Bharata's system of twenty-two *śrutis*, with the exception that this system recognizes twenty-three note-positions (although not technically speaking twenty-three, but only twenty-two "*śrutis*") nor is it really necessary to do so, except to the scholar who wishes to be troublesome. Therefore, I shall not attempt to do so here, begging the reader's sympathy. Rather, we turn to the consideration of another problem.

The tonal space of the octaves is a phenomenon which is describeable mathematically not by an arithmetic series (1,2,3,4,...etc.) but by the geometrical series (1,2,4,8,16,...etc.). To "add" one note's interval to another note's requires not addition but multiplication of the corresponding ratios. Similarly "subtraction" of intervals requires the division of corresponding ratios. In order to simplify this process, we usually resort to using a logarithmic system such as the system invented by the French physicist, Savart, wherein values of intervals are expressed in "savarts" which is nothing more than the values expressed by the mantissa in base 10 logarithms multiplied by 1,000 to remove the decimal. Thus an octave has the value, approximately, 301 savarts. (An octave is twice the value of the base, hence $2/1$. $\log_{10} 2 = .30103$, times 1,000 = 301.03 savarts.)

This use of base ten logarithms, while it is perfectly adequate to solve the algebraic problem of simplifying the manipulation of ratios in the comparison of note-intervals is a poor solution to the graphic problem of representing the division of the octave on paper. The obvious cyclic property of octaves in which each note is "repeated" every eight steps invites a graphic representation of the octave and its division by means of a circle. But in order to represent the octave as a circle, it is necessary to be able to express the values of the intervals in terms of a ruler whose basic configuration of units likewise repeats every octave.

From this point of view, the inadequacies of the savart system (to say nothing of Ellis' cents system which is even worse) become obvious. A new octave starts after, approximately, 301 savarts, but because we adhere to a decimal base, the characteristic of the logarithm changes after 999+ savarts. Within that many savarts, we have more than three octaves. How much better would be a system of valuation, based on logarithms all right, but in which the full range of the mantissa, .000—.999, described exactly no more nor no less than one complete octave. Then the number of the *n*th octave above the base would be reflected by the characteristic (*n*) of the logarithm. We should then have a ruler which when layed out on a circle would exactly match the beginning with the end. The serpent could bite his tail, so to speak, without having to swallow any of it.

Such a system of measurement is available by making use of base 2 logarithms. We have accordingly designed our graph which appears at the end of this paper according to this rule, taking advantage of the fact that the logarithm in base 2 naturally corresponds to the phenomenon of the octave. Another advantage of this method is that we end up with a value of the intervals in terms of percentages which corresponds to the way in which we perceive intervals. The actual frequency of the fourth octave is 8 times the frequency of the base although it only "seems" to be four times as high as the base, etc.

The fact that the octave can be represented on a circle greatly simplifies the study of the relationships between the intervals of a scale and the intervals of its plagal modes, or *mūrccchanās*. (If C to upper c is a scale, taking the white keys of a piano for example, then D to d is the scale of the D *mūrccchanā*, E to e the scale of the E *mūrccchanā*, etc.) Once we succeed in representing the octave as a circle, we can construct a simple device similar to a circular slide rule which will show interval relationships among *mūrccchanās* by placing a wheel marked according to the intervals of the basic scale against the background of the division of the octave.

In this paper we present the reader with such a device which can be made by cutting out the scale or *grāma* wheel, cutting out the wheel of the division of the octave by *śrutis*, mounting both on a suitable stiff backing and pinning the *grāma* wheel to the *śruti* wheel. The name of the "spoke" of the *grāma*

wheel which matches with the Sa of the *śruti* wheel indicates the name of the *mūrchanā* of the basic scale. Its intervals can be read off by following the spokes to the indicated *śrutis*.

One obvious question is why select one scale as "basic" in favor of another? There are two answers to this question. One is based on acoustical considerations, the other on certain traditional considerations. Scales in general can be based on usually no fewer than five notes, nor no more than twelve. We choose to make our "basic" or parent (*janaka*) scale of twelve notes since a twelve note scale is "full" there being no thirteenth note possible in an octave, except the repetition of the base an octave higher. The advantage of doing this is that all of the mutual consonances (i.e., perfect fourths and fifths) within the scale are displayed. Any scale of less than twelve notes can be viewed as merely transilient (*janya*), made by dropping certain intervals from the parent chromatic scale type.

But still, why pick one chromatic scale rather than another? From the system of 23 intervals, by selecting now the lower, now the upper alternative of each of the notes, it is possible to construct 2048 different chromatic scales. How to decide which of these is "basic?"

We make this selection according to the following criteria. First, it must be recognized that it is impossible to construct a chromatic scale in which every note has both a perfect fourth and perfect fifth in the scale. Were this not so, we would never have been forced into the compromise of equal temperament in the West. At the very minimum there must be at least one "modal" fifth or fourth in the chromatic scale. A modal fifth or fourth, known to Renaissance musicians as a "wolf" tone because of its "howl" is a fifth too small by a comma, or a fourth too large by a comma.

Now while every chromatic scale must have one such "wolf" (and its inversion, of course), a chromatic scale can have up to as many as 11 such "wolf tones." We simply assume that the fewer the wolf tones, the better the scale. On this basis, we can eliminate 2036 chromatic scales from our consideration leaving twelve scales remaining. In each of these twelve scales, there is only one imperfect fifth.

Pick any one of these scales as basic and the remaining eleven show up as *mūrchanās* of the one made

basic. Hence our second criterion is that the basic scale should be a scale which stands at the limit of whatever series these scales make up.

Now consider the following tuning experiment which will be familiar to anyone who has tried to tune his piano without knowledge of the piano tuner's art. Suppose I try to tune my piano according to the cycle of fifths starting from "C" staying within the limits of one octave. The last note I tune in the series will be "F". When I check that "F" against the original "C" I discover the wolf tone. I have tuned, then, one of the twelve maximally consonant chromatic scales in which the last fifth of the series forms an imperfect relation to the base. But suppose, not knowing the piano tuner's dilemma (nor his art) I think to myself, "This cannot be!" and retune that "F" to match perfectly with "C". I shall then discover that I have shifted the wolf from "F—C" to "B^b—F". I shall have another of the maximally consonant chromatic scales nevertheless. Still, if I am not satisfied, I can retune the "B^b", but this will only again shift the modal fifth now to "E^b—B^b". In this way, I can continue until I reach the last step going backwards in the cycle of fifths readjusting each time until I have lowered "G" by a comma, at which point, I will again have a modal fifth between this "G" and my original base, "C". As I began having a chromatic scale in which every note was determined by the cycle of fifths, I shall end up, working backwards in this manner, with a scale in which every note was determined by the cycle of fourths (or descending fifths). Beyond either of these limits I cannot go without tampering with my original base, "C".

It should be clear, that taking the idea of the limiting scale as a criterion for determining a basic chromatic scale, there are two such basic scales: one having all notes in Sa-Pa *bhāva* (the cycle of fifths) and the other having all notes in Sa-Ma *bhāva* (the cycle of fourths).

Bharata's *Nāṭyaśāstra* (c. 1st Century A.D.) mentions two basic scales. One is called the *Ma-grāma*, the other is called the *Sa-grāma*. Since the medieval period scholars have tended to accept the *Sa-grāma* as the basic (*śuddha*) scale of Bharata. But both are of equal importance in the *Nāṭyaśāstra*. They are virtually the same, each composed of nine notes, differing only in the fact that the fifth (Pa

or "G") of the Ma-grāma forms a modal fifth with Sa.

Suppose we had a board in which twelve sets of holes were drilled, and in addition, we had a piece of wood in which nine pegs were mounted. If we found that our set of pegs would fit into only one set of holes, we would have reason to suspect that, even though we had three less pegs than holes, there was some kind of systematic relationship between our set of holes and our set of pegs.

It is on the basis of this analogy, backed up by certain additional considerations which we cannot touch here that we (i) add the last, and the "traditional" criterion for selecting a "basic" chromatic scale, and (ii) at the same time offer a way to extend Bharata's principles, but extend them consistently, in order to account for several of those modern rāgas which are beyond the literal purview of Bharata's system.

We do not want to limit ourselves to following the ancient system solely, but we do hope to take a hint from the ancient ṛṣis. Therefore, we take as our basic scale only that scale of the set of twelve maximally consistent chromatic scales into which Bharata's Ma-grāma "fits." We do not take that scale into which the Sa-grāma fits because it does not stand at the position of a limit. In our system, the Sa-grāma fits into the Ma-mūrccchanā, as according to Bharata it should, of the basic scale which we choose to call the "Ma-grāma" chromatic scale.

This chromatic scale type contains the notes whose intervals fall on or near within one part in six-hundred of the śruti determined by one complete round of the cycle of fourths. Of all pairs of śrutis for each note, the notes of this scale all stand upon the lower śruti—barring the difference of the *śikṣā* (the "one part in six-hundred") which has considerable theoretical interest, but little practical significance.

On the basis of this scale type, it is possible to classify the remaining eleven of the most consonant chromatic scales according to their order manifest in the mūrccchanās of the one basic scale. Briefly stated, that order may be expressed as follows. If taking the "Ma-grāma" chromatic scale as basic, the mūrccchanās of the notes are taking in the order of the cycle of fourths, the scales which result from those mūrccchanās show a step by step cumulative increase in the śruti value of the notes of the scale: each step through the chain of the cycle of fourths by

mūrccchanā produces a chromatic scale the notes of which gain a śruti according to the order of the cycle of fifths.

In the Sa mūrccchanā the lower of the pairs of śrutis is always taken. Therefore, the Sa mūrccchanā possesses the most minor of these chromatic scales. As the order of the mūrccchanās progresses following the order of the cycle of fourths, the notes in the order of the cycle of fifths' series are each raised by a comma. From step 1 to step 6 of the chain, this change affects only the major notes which change from Harmonic, or weaker, to Pythagorean, or stronger. From step 7 to step 12 inclusive, the change affects only the minor notes which change from Pythagorean or stronger, to Harmonic or weaker. In the last mūrccchanā of the series, the higher of the pairs of śrutis is always taken, hence this mūrccchanā contains the most major of these chromatic scales.

From the point of view of aesthetic flavor (*rasa*) the first mūrccchanā of the series describes the situation of strong minors plus weak majors which progressively become stronger until at step six, strong minors are paired with strong majors. After rank 7, the situation moves by degrees to a gradual weakening of the minors until the 12th step is reached which contains all majors strong and all minors weak. The mūrccchanās from the middle region of this chain are the most active, containing the most Pythagorean intervals in their scales, and therefore the intervals of the higher ratios. Higher ratios imply higher tension levels of resonance. Aesthetically, then, the cool notes (the minors) are at their coldest precisely when the warm notes (the majors) are at their hottest.

Exactly this type of change occurs during the period of the day when at *sandhiprakāśa* the cool of the night yields to the heat of the day or the heat of the day yields to the cool of the night. It is quite consistent with the order of Nature that, as our studies seem to show, the *sandhiprakāśa* rāgas take their tunings from the mūrccchanās near the sixth or seventh step of the chain.

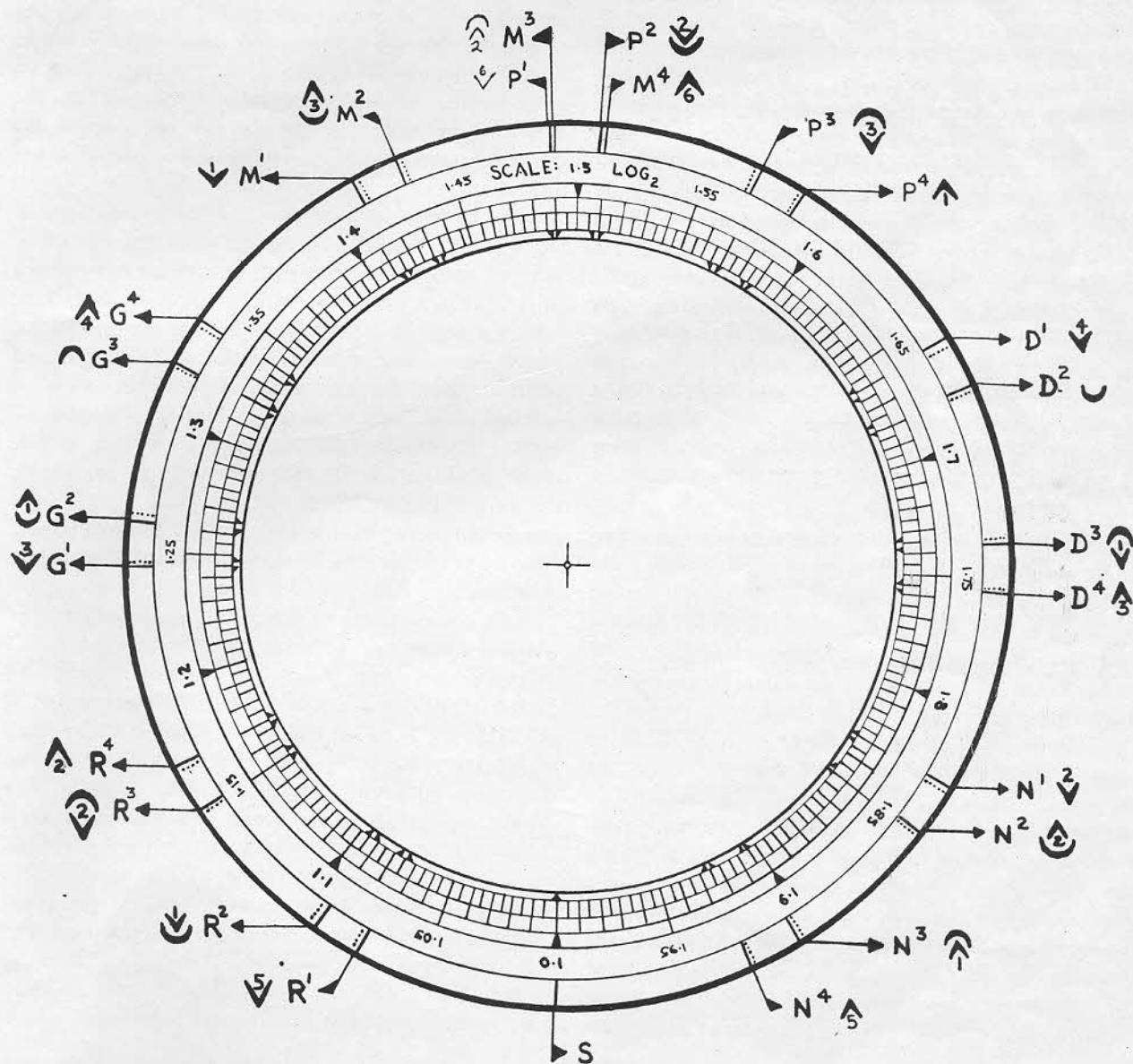
As evening *sandhiprakāśa* may be said to represent the hottest part of the night from which point on the night becomes cooler, so morning *sandhiprakāśa* may be said to represent the coolest part of the morning from which point on it becomes warmer. Analogous-

ly, the sixth and seventh steps contain both the hottest of the majors, which by degrees "cool" moving from step six or seven backwards, and also the coldest minors which progressively "warm" from step six or seven forwards. Thus, we should expect that *rāgas* of the night will evolve from scales that, as the night goes on are born of the chromatic *mūrchanās* progressively nearing step one. And, so too, we should expect that the *rāgas* of the morning will evolve out of scales, that as the morning develops are born out of the scales of the *mūrchanās* progressing towards the twelfth step. But this is an

extended topic for investigation and can only be hinted at here.

Be that as it may, these and other relationships among the consistent chromatic scales or their transilient lesser note versions can be easily demonstrated and examined by means of the graphic tool given here.

A word of mention is in order on the layout of this device. The *śruti* positions marked on the graph are named after the Indian scale names. Sa, Re, Ga, Ma, Pa, Dha, Ni, in Indian names is equivalent to the solfeggio Do, Re, Me, Fa, Sol, La, Ti of the



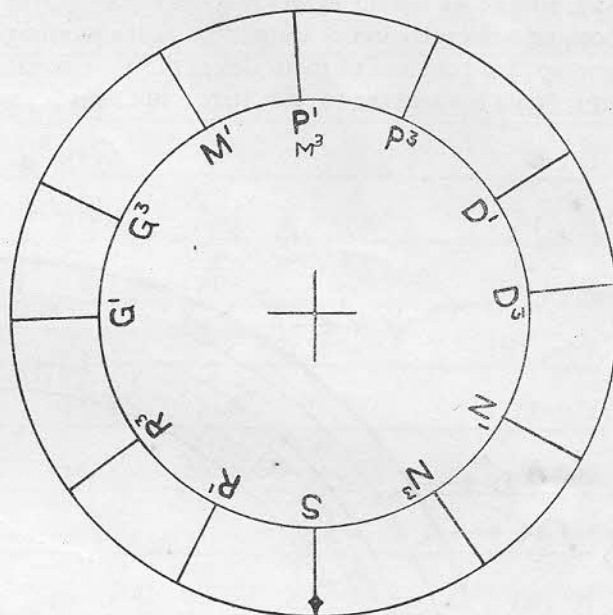
Graph in Log₂ Units of the System of Śrutis

West. Each *śruti* is given a number as well as a name indicating its relative rank within the degree specified by the note name. Along side each name is a tuning symbol which describes how to tune the interval. The idea for this system of symbols belongs to Jacques Dudon. I have borrowed and adapted it because of its simplicity. The system of symbols should be read as follows :

- read ♠ as the nth ascending fifth from Sa.
- read ♡ as the nth descending fifth from Sa.
- read ~ as a major third ascending from Sa.
- read ∪ as a major third descending from Sa.
- read ♠ or ♡ as the nth ascending, or descending fifth from the major third above Sa.
- read ♠ or ♡ as the nth ascending, or descending fifth from the major third below Sa.

Another fact that needs mention is that although I have argued the advantage of using base 2 logarithms to measure intervals, due to the difficulty of obtaining Tables of base 2 logarithms, I have retained the traditional savarts, converting them into degrees in order to construct this graph. There is drawn onto the graph a scale in base 2 logarithmic units, but due to the difficulties of laying this out by hand, the reader should be warned against trying to read the scale too exactly. We wish to express our gratitude for help in this latter task to John Deyell who patiently drew the graph presented here.

A set of tables is also given here showing the values of the intervals of the cycles of fifths and fourths and their Harmonic replacements. In the graph, those positions which are determined by the simpler ratios are marked in solid lines, and those whose ratios are more complex are marked in dotted lines.



The "Ma-Grama" Chromatic Scale

Note:—Due to non-availability of complex diacritical marks necessary for the technical terminology of this highly specialized paper, we regret it has not been possible to mark them as they should have been, our apologies to the author and the readers.

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TABLE I
SERIES A : CYCLE OF FIFTHS

P=Pythagorean
H=Harmonic
P>H

Note	Tune	Ratio	Integer Ratio	Decimal Ratio	Savarts	Degrees	C.P.S. Freq	Ratio	Differences Deg.	Freq
SA	—	$\frac{1}{1}$	$\frac{1}{1}$	1.0000	0	0	512	—	—	—
PA ₄ (P)	$\hat{1}$	$\frac{3^1}{2^1}$	$\frac{3}{2}$	1.5000	176.1	210.6	768.00	0.0017	0.5	0.86
PA ₄ (H)	$\hat{2}$	$\frac{8 \cdot 2^{11}}{5 \cdot 3^7}$	$\frac{16834}{10935}$	1.4983	175.7	210.1	767.14			
Re ₄ (P)	$\hat{2}$	$\frac{3^2}{2^3}$	$\frac{9}{8}$	1.1250	51.1	61.2	576.00	0.0013	0.3	0.65
Re ₄ (H)	$\hat{3}$	$\frac{8 \cdot 2^9}{5 \cdot 3^6}$	$\frac{4096}{3645}$	1.1237	50.6	60.5	575.35			
DhA ₄ (P)	$\hat{3}$	$\frac{3^3}{2^4}$	$\frac{27}{16}$	1.6875	227.2	271.7	864.00	0.0019	0.4	0.97
DhA ₄ (H)	$\hat{4}$	$\frac{8 \cdot 2^8}{5 \cdot 3^5}$	$\frac{2048}{1215}$	1.6856	226.8	271.3	863.03			
GA ₄ (P)	$\hat{4}$	$\frac{3^4}{2^6}$	$\frac{81}{64}$	1.2656	102.3	122.3	648.00	0.0014	0.5	0.73
GA ₄ (H)	$\hat{5}$	$\frac{8 \cdot 2^8}{5 \cdot 3^4}$	$\frac{512}{405}$	1.2642	101.8	121.8	647.27			
Ni ₄ (P)	$\hat{5}$	$\frac{3^5}{2^7}$	$\frac{243}{128}$	1.8984	278.5	333.1	972.00	0.0021	0.7	1.10
Ni ₄ (H)	$\hat{6}$	$\frac{8 \cdot 2^5}{5 \cdot 3^3}$	$\frac{256}{135}$	1.8963	277.9	332.4	970.90			
MA ₄ (P)	$\hat{6}$	$\frac{3^6}{2^9}$	$\frac{729}{512}$	1.4238	153.4	183.5	729.00	0.0016	0.6	0.72
PA ₂ (H)	$\hat{7}$	$\frac{8 \cdot 2^3}{5 \cdot 3^2}$	$\frac{64}{45}$	1.4222	152.9	182.9	728.18			
Re ₂ (P)	$\hat{7}$	$\frac{3^7}{2^{11}}$	$\frac{2187}{2048}$	1.0679	28.6	34.2	546.75	0.0012	0.6	0.61
Re ₂ (H)	$\hat{8}$	$\frac{8 \cdot 2}{5 \cdot 3}$	$\frac{16}{15}$	1.0667	28.1	33.6	546.14			
Dha ₂ (P)	$\hat{8}$	$\frac{3^8}{2^{12}}$	$\frac{6561}{4096}$	1.6018	204.6	244.7	820.13	0.0018	0.6	0.93
Dha ₂ (H)	$\hat{9}$	$\frac{8}{5}$	$\frac{8}{5}$	1.6000	204.1	244.1	819.20			
GA ₂ (P)	$\hat{9}$	$\frac{3^9}{2^{14}}$	$\frac{19683}{16384}$	1.2014	79.7	95.3	615.09	0.0014	0.6	0.69
GA ₂ (H)	$\hat{10}$	$\frac{8 \cdot 3^1}{5 \cdot 2^2}$	$\frac{6}{5}$	1.2000	79.2	94.7	614.40			
Ni ₂ (P)	$\hat{10}$	$\frac{3^{10}}{2^{15}}$	$\frac{59049}{32768}$	1.8020	255.8	305.9	922.64	0.0020	0.6	1.04
Ni ₂ (H)	$\hat{11}$	$\frac{8 \cdot 3^2}{5 \cdot 2^3}$	$\frac{9}{5}$	1.8000	255.3	305.3	921.60			
MA ₂ (P)	$\hat{11}$	$\frac{3^{11}}{2^{17}}$	$\frac{177147}{131072}$	1.3515	130.7	156.3	691.98	0.0015	0.5	0.78
MA ₂ (H)	$\hat{12}$	$\frac{8 \cdot 3^3}{5 \cdot 2^5}$	$\frac{27}{20}$	1.3500	130.3	155.8	691.29			

TABLE II

SERIES B: THE CYCLE OF FOURTHS

P=Pythagorean

H=Harmonic

H > P

Note	Tune	Ratio	Integer Ratio	Decimal Ratio	Savarts	Degrees	Freq.	Ratio	Deg.	Freq.
SA		$\frac{2}{1}$	$\frac{2}{1}$	2.0000	.3010	360	1024	—	—	—
MA ¹ (P)	↓	$\frac{2^2}{3^1}$	$\frac{4}{3}$	1.3333	124.9	149.4	682.66			
MA ¹ (H)	↗	$\frac{5 \cdot 3^7}{4 \cdot 2^{11}}$	$\frac{10935}{8192}$	1.3348	125.5	150.1	683.44	0.0015	0.7	0.78
Ni ¹ (P)	↘	$\frac{2^4}{3^2}$	$\frac{16}{9}$	1.7778	249.9	298.9	910.23			
Ni ¹ (H)	↖	$\frac{5 \cdot 3^6}{4 \cdot 2^9}$	$\frac{3645}{2048}$	1.7798	250.3	299.4	911.25	0.0020	0.5	1.02
GA ¹ (P)	↘	$\frac{2^5}{3^3}$	$\frac{32}{27}$	1.1852	73.9	88.4	606.82			
GA ¹ (H)	↖	$\frac{5 \cdot 3^5}{4 \cdot 2^8}$	$\frac{1215}{1024}$	1.1865	74.3	88.9	607.50	0.0013	0.4	0.68
Dha ¹ (P)	↓	$\frac{2^7}{3^4}$	$\frac{128}{81}$	1.5802	198.7	237.6	809.08			
Dha ¹ (H)	↖	$\frac{5 \cdot 3^4}{4 \cdot 2^6}$	$\frac{405}{256}$	1.5820	199.3	238.4	810.00	0.0018	0.8	0.92
Re ¹ (P)	↘	$\frac{2^8}{3^5}$	$\frac{256}{243}$	1.0535	22.7	27.2	539.39			
Re ¹ (H)	↖	$\frac{5 \cdot 3^3}{4 \cdot 2^5}$	$\frac{135}{128}$	1.0547	23.2	27.7	540.00	0.0012	0.5	0.61
P ¹ (P)	↘	$\frac{2^{10}}{3^6}$	$\frac{1024}{729}$	1.4047	147.6	176.5	719.19			
MA ³ (H)	↖	$\frac{5 \cdot 3^2}{4 \cdot 2^3}$	$\frac{45}{32}$	1.4062	147.9	176.9	720.00	0.0015	0.4	0.81
Ni ³ (P)	↘	$\frac{2^{12}}{3^7}$	$\frac{4096}{2187}$	1.8729	272.5	325.9	958.92			
Ni ³ (H)	↖	$\frac{5 \cdot 3}{4 \cdot 2}$	$\frac{15}{8}$	1.8750	273.0	326.5	960.00	0.0021	0.6	1.08
GA ³ (P)	↘	$\frac{2^{13}}{3^8}$	$\frac{8192}{6561}$	1.2486	96.4	115.3	639.28			
GA ³ (H)	↖	$\frac{5}{4}$	$\frac{5}{4}$	1.2500	96.9	115.9	640.00	0.0014	0.6	0.72
Dha ³ (P)	↘	$\frac{2^{15}}{3^9}$	$\frac{32768}{19683}$	1.6648	221.4	264.8	852.37			
Dha ³ (H)	↖	$\frac{5 \cdot 2^2}{4 \cdot 3^1}$	$\frac{5}{3}$	1.6667	221.8	265.3	853.34	0.0019	0.5	0.97
Re ³ (P)	↘	$\frac{2^{16}}{3^{10}}$	$\frac{65536}{59049}$	1.1099	45.3	54.2	568.27			
Re ³ (H)	↖	$\frac{5 \cdot 2^3}{4 \cdot 3^2}$	$\frac{10}{9}$	1.1111	45.7	54.7	568.09	0.0012	0.5	0.16
PA ³ (P)	↘	$\frac{2^{18}}{3^{11}}$	$\frac{262144}{177147}$	1.4798	170.2	203.6	757.66			
PA ³ (H)	↖	$\frac{5 \cdot 2^5}{4 \cdot 3^3}$	$\frac{40}{27}$	1.4815	170.7	204.2	758.51	0.0017	0.6	0.85